# Supersymmetry breaking, conserved charges and stability in $N = 1$ Super KdV

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**Abstract**

We analyse the non-abelian algebra and the supersymmetric cohomology associated to the local and non-local conserved charges of N=1 SKdV under Poisson brackets. We then consider the breaking of the supersymmetry and obtain an integrable model in terms of Clifford algebra valued fields. We discuss the remaining conserved charges of the new system and the stability of the solitonic solutions.

Keywords: supersymmetric models, integrable systems, conservation laws, nonlinear dynamics of solitons, partial differential equations

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## 1 Introduction

Local and non-local conserved charges are relevant in order to analyze perturbative and non-perturbative aspects of a field theory. It is known that the local conserved charges of $N = 1$ SKdV are related to the superconformal algebra. It is then interesting to look for an extension of the abelian algebra, spannned by the local conserved charges of $N = 1$ SKdV, in terms of the non-local conserved charges. We obtain such extension as a non-abelian algebra realized in terms of Poisson brackets and discuss its relation to the supersymmetric structure of $N = 1$ SKdV. We then consider the susy breaking on

the symmetries associated to the higher dimensional conserved charges. The resulting system, although has less symmetries than the original SKdV one, has solitonic solutions with nice stability properties.

# 2 Non-abelian algebra of conserved charges and supersymmetric cohomology for $N = 1$ SKdV

The susy $N = 1$ Korteweg-de Vries equation in superfield form is given by

$$\Phi_t = D^6\Phi + 3D^2 (\Phi D\Phi), \tag{1}$$

where $\Phi = \xi + \theta u$ and $\xi, u$ are fields with values on the odd and even parts respectively of a given Grassmann algebra $\Lambda$ with $n$ generators and a distinguished one $\theta$, and $D = \partial_\theta + \theta\partial_x$ is the covariant derivative with the property $D^2 = \partial_x$ acting on superfields.

In components (1) is equivalent to

$$\begin{aligned} u_t &= u''' + 6uu' - 3\xi\xi'' \\ \xi_t &= \xi''' + 3(\xi u)'. \end{aligned} \tag{2}$$

The system (2) is integrable in the sense that it admits an infinite number of local conserved charges. It is invariant under the supersymmetric transformation defined by $\delta\Phi = \eta Q\Phi$, where $\eta$ is an odd parameter and $Q = -\partial_\theta + \theta\partial_x$. It is also known to have an infinite sequence of odd and even non-local conserved charges. They can be obtained using simultaneously an auxiliary equation, the so called super-Gardner equation

$$\chi_t = D^6\chi + 3D^2(\chi D\chi) - \varepsilon^2 3(D\chi)D^2(\chi D\chi) \tag{3}$$

and the super-Gardner transformation $\Phi = \chi + \varepsilon D^2\chi - \varepsilon^2\chi D\chi$, relating (1) with (3) and specifically solutions between them. The infinite sequence of local conserved charges of the $N = 1$ Super KdV equation were obtained by Mathieu [1] from the one conserved charge of the super-Gardner equation

$$G_1 = \int dx d\theta \chi = \sum_{n=0} \epsilon^{2n} H_{2n+1}, \tag{4}$$

using the inverse Gardner transformation. The infinite sequence of fermionic non-local conserved charges were obtained by Dargis and Mathieu [2] and Kersten [3]. This infinite sequence can be obtained also from one non-local conserved charge of the super-Gardner equation [4]

$$G^{NL}_{\frac{1}{2}} = \int dx d\theta \left( \frac{\exp(\epsilon D^{-1}\chi) - 1}{\epsilon} \right) = \sum_{n=0} \epsilon^n H^{NL}_{n+\frac{1}{2}}. \tag{5}$$

And finally, the infinite sequence of bosonic non-local conserved charges was derived [5] from the non-local conserved charge for super-Gardner given by

$$H_G = \frac{1}{2}\int dxd\theta D^{-1}\left\{D\left[\frac{\exp\left(\epsilon D^{-1}\chi\right)-1}{\epsilon}\right]D^{-1}\left[\frac{\exp\left(-\epsilon D^{-1}\chi\right)-1}{\epsilon}\right]\right\} = \sum_{n=0}\epsilon^n H^{NL}_{n+1}. \tag{6}$$

In (4),(5) and (6) the subscripts of the right members denote the dimension of the respective conserved charges.

It is interesting to notice that the Gardner deformation for $N = 2$ supersymmetric $a = 4$ KdV equation has also been recently solved [6, 7].

Given a conserved charge for equation (1) of the form $H = \int dxd\theta h, h \in C^\infty_I$, where $C^\infty_I$ is the ring of integrable superfields, we define

$$\delta_Q H := \int dxd\theta Q h, \tag{7}$$

where $\delta_Q$ satisfies $\delta_Q\delta_Q = 0$. (7) defines a cohomology on the conserved charges of super-KdV [5]. In fact, for example we have

$$\delta_Q H_{2n+1} = 0, n = 0, 1, \ldots, \tag{8}$$

hence $H_{2n+1}$ must be $\delta_Q$ of some other conserved charge, in particular for $n = 1$ we have

$$\delta_Q H^{NL}_{\frac{1}{2}} = H_1. \tag{9}$$

In the same way we obtain a relation between the non-local odd and even conserved charges, for example

$$\delta_Q H^{NL}_1 = H^{NL}_{\frac{3}{2}} - \frac{1}{2}H_1 H^{NL}_{\frac{1}{2}}. \tag{10}$$

The general formula is

$$\sum_{n=0}\epsilon^n\delta_Q H^{NL}_{n+1} = \sum_{n=0}\epsilon^{2n}H^{NL}_{2n+\frac{3}{2}} - \frac{1}{2\epsilon}\left[\exp\left(\sum_{n=0}\epsilon^{2n+1}H_{2n+1}\right)-1\right]\left[\sum_{n=0}(-\epsilon)^n H^{NL}_{n+\frac{1}{2}}\right]. \tag{11}$$

We then consider the Poisson bracket of superfields $\Phi$ at two different points in superspace as given in [1]:

$$\{\Phi(x_1,\theta_1), \Phi(x_2,\theta_2)\} = P_1\Delta, \tag{12}$$

where $P_1 = D_1^5 + 3\Phi_1 D_1^2 + (D_1\Phi_1)D_1 + 2(D_1^2\Phi_1)$ and $\Delta = \delta(x_1 - x_2)(\theta_1 - \theta_2)$. With this bracket, using the supersymmetric cohomology and after some calculations, it is possible to derive the Poisson algebra of local and non-local conserved charges, in particular we

obtain the following relations:

$$\left\{H^{NL}_{\frac{1}{2}}, H^{NL}_{\frac{1}{2}}\right\} = H_1, \tag{13}$$

$$\left\{H^{NL}_{\frac{1}{2}}, H^{NL}_{\frac{3}{2}}\right\} = \frac{1}{2}H_1^2, \tag{14}$$

$$\left\{H^{NL}_{\frac{1}{2}}, H^{NL}_{1}\right\} = -\frac{1}{2}H_1 H^{NL}_{\frac{1}{2}} + H^{NL}_{\frac{3}{2}}, \tag{15}$$

and

$$\left\{H^{NL}_{\frac{3}{2}}, H^{NL}_{\frac{3}{2}}\right\} = -H_3 + \frac{1}{3}(H_1)^3. \tag{16}$$

This gives the first steps in order to construct the complete non-abelian algebra of conserved charges for $N = 1$ super-KdV equation. This problem is under development.

The local and non-local conserved charges are related to symmetries of the $N = 1$ SKdV equation. We are now interested in considering the breaking of supersymmetry which will induce a breaking of the symmetries associated with the conserved charges.

# 3 Supersymmetry breaking of SKdV and stability of the ground state and the one-soliton solutions

In order to break the supersymmetry we consider the system [8]

$$\begin{aligned} u_t &= -u''' - uu' - \tfrac{1}{4}\left(\mathcal{P}\left(\xi\bar{\xi}\right)\right)' \\ \xi_t &= -\xi''' - \tfrac{1}{2}(\xi u)', \end{aligned} \tag{17}$$

where the fields $u$ and $\xi$ takes values on a Clifford algebra instead of being Grassmann algebra valued. In order to compare with the stability analysis of KdV in Benjamin [9] we have redefined the $u$ and $\xi$ fields in (2) with convenient factors. We thus take $u$ to be a real valued field while $\xi$ to be an expansion in terms of the generators $e_i, i = 1, \ldots$ of the Clifford algebra:

$$\xi = \sum_i \varphi_i e_i + \sum_{ij} \varphi_{ij} e_i e_j + \sum_{ijk} \varphi_{ijk} e_i e_j e_k + \cdots \tag{18}$$

where

$$e_i e_j + e_j e_i = -2\delta_{ij} \tag{19}$$

and $\varphi_i, \varphi_{ij}, \varphi_{ijk}, \ldots$ are real valued functions. We define $\bar{\xi} = \sum_{i=1}^{\infty} \varphi_i \bar{e}_i + \sum_{ij} \varphi_{ij} \bar{e}_j \bar{e}_i + \sum_{ijk} \varphi_{ijk} \bar{e}_k \bar{e}_j \bar{e}_i + \cdots$ where $\bar{e}_i = -e_i$. We denote as in superfield notation the body of the expansion those terms associated with the identity generator and the soul the remaining ones. Consequently the body of $\xi\bar{\xi}$, denoted by $\mathcal{P}(\xi\bar{\xi})$, is equal to $\Sigma_i \varphi_i^2 + \Sigma_{ij} \varphi_{ij}^2 + \Sigma_{ijk} \varphi_{ijk}^2 + \cdots$ In what follows, without loss of generality, we rewrite $\mathcal{P}(\xi\bar{\xi}) =$

$\Sigma_i\varphi_i^2+\Sigma_{ij}\varphi_{ij}^2+\Sigma_{ijk}\varphi_{ijk}^2+\cdots$ simply as $\mathcal{P}(\xi\bar{\xi})=\Sigma_i\varphi_i^2$. The system (17) is not invariant under supersymmetric transformations and has not a conserved charge of dimension 7, that is there is no analogue of $H_7$ in SKdV or KdV systems but has the following conserved charges:

$$\begin{array}{l}\hat{H}_{\frac{1}{2}}=\int_{-\infty}^{+\infty}\xi dx,\quad \hat{H}_1=\int_{-\infty}^{+\infty}udx\\ V\equiv\hat{H}_3=\frac{1}{2}\int_{-\infty}^{+\infty}\left(u^2+\mathcal{P}\left(\xi\bar{\xi}\right)\right)dx,\\ M\equiv\hat{H}_5=\frac{1}{2}\int_{-\infty}^{+\infty}\left(-\frac{1}{3}u^3-\frac{1}{2}u\mathcal{P}\left(\xi\bar{\xi}\right)+\left(u'\right)^2+\mathcal{P}\left(\xi'\bar{\xi}'\right)\right)dx.\end{array}\tag{20}$$

It is interesting to remark that the following non-local conserved charge of Super KdV [5] is also a non-local conserved charge for the system (17), in terms of the Clifford algebra valued field $\xi$,

$$\int_{-\infty}^{\infty}\xi(x)\int_{-\infty}^{x}\xi(s)dsdx.$$

However the non-local conserved charges of Super KdV in [2] are not conserved by the system (17). For example,

$$\int_{-\infty}^{\infty}u(x)\int_{-\infty}^{x}\xi(s)dsdx$$

is not conserved by (17).

We now consider the stability of solutions for the system (17) in the sense of Liapunov [8]. In particular we take the same definition as in [9]: $(\hat{u},\hat{\xi})$, a solution of (17), is stable if given $\epsilon$ there exists $\delta$ such that for any solution $(u,\xi)$ of (17), satisfying at $t=0$

$$d_I\left[\left(u,\xi\right),\left(\hat{u},\hat{\xi}\right)\right]<\delta\tag{21}$$

then

$$d_{II}\left[\left(u,\xi\right),\left(\hat{u},\hat{\xi}\right)\right]<\epsilon\tag{22}$$

for all $t\geq 0$.

$d_I$ and $d_{II}$ denote two distances to be defined.

We denote by $\|\ \|_{H_1}$ the Sobolev norm

$$\|(u,\xi)\|_{H_1}{}^2=\int_{-\infty}^{+\infty}\left[\left(u^2+\Sigma_{i=0}^{\infty}\varphi_i^2\right)+\left(u'^2+\Sigma_{i=0}^{\infty}\varphi_i'^2\right)\right]dx.\tag{23}$$

It can be proved that

$$\|(u,\xi)\|_{H_1}{}^2\leq V+M+\frac{1}{\sqrt{2}}V\|(u,\xi)\|_{H_1},\tag{24}$$

and this gives a priori bound for solutions to the system (17). The stability of the ground state solution $\hat{u}=0,\hat{\xi}=0$ is a direct consequence of (24) taking $d_I$ and $d_{II}$ to be the Sobolev norm $\|(u-\hat{u},\xi-\hat{\xi})\|_{H_1}$.

The system (17) has also solitonic solutions, for example: $u(x,t) \equiv \phi(x,t) = 3\mathcal{C}\frac{1}{\cosh^2(z)}, z \equiv \frac{1}{2}\mathcal{C}^{\frac{1}{2}}(x - \mathcal{C}t), \xi(x,t) = 0$. $\phi(x,t)$ is the one-soliton solution of KdV equation. We now consider the stability of the one-soliton solution $\hat{u} = \phi, \hat{\xi} = 0$. The proof of stability is based on estimates for the $u$ field which are analogous to the one presented in [9, 10] while a new argument will be given for the $\xi$ field. The distances we will use are

$$d_I\left[(u_1, \xi_1), (u_2, \xi_2)\right] = \parallel (u_1 - u_2, \xi_1 - \xi_2) \parallel_{H_1} \tag{25}$$

$$d_{II}\left[(u_1, \xi_1), (u_2, \xi_2)\right] = \inf_{\tau} \parallel (\tau u_1 - u_2, \xi_1 - \xi_2) \parallel_{H_1} \tag{26}$$

where $\tau u_1$ denotes the group of translations along the $x$-axis. $d_{II}$ is a distance on a metric space obtained by identifying the translations of each $u \in H_1(\mathbb{R})$. $d_{II}$ is related to a stability in the sense that a solution $u$ remains close to $\hat{u} = \phi$ only in shape but not necessarily in position.

We first assume that

$$V(u, \xi) = V(\hat{u}, \hat{\xi}) = V(\phi, 0) \tag{27}$$

and

$$\int_{-\infty}^{+\infty} \xi dx = \int_{-\infty}^{+\infty} \hat{\xi} dx = 0 \tag{28}$$

and at the end of the argument we relax this conditions to get the most general formulation of the stability problem. It is then possible to prove, after long reasoning that

$$\Delta M \equiv M(u, \xi) - M(\phi, 0) \leq \left[\max(1, \mathcal{C}) + \frac{1}{3\sqrt{2}}\delta\right] \parallel (h, \xi) \parallel^2_{H_1} \tag{29}$$

and that

$$\Delta M \geq \frac{1}{6} l \{d_{II}\left[(u, \xi), (\phi, 0)\right]\}^2, \tag{30}$$

and these two inequalities, using that $\Delta M$ is a conserved quantity, are sufficient to prove the stability of the one-soliton solution for the system (17). To relax the conditions (27) and (28) we use essentially the triangle inequality.

# 4 Conclusions

We discussed the algebraic structure of the local and non-local conserved charges of $N = 1$ SKdV. We presented a rigorous construction of the Poisson brackets of these non-local conserved charges, this is in general an open problem in field theory.

The non-abelian algebra has been worked out partially, we hope to provide the complete algebraic structure in a future work. Besides we considered the supersymmetry

breaking of $N = 1$ SKdV, this is always an interesting problem to study. The resulting system is integrable, in the sense that it has solitonic solutions, but, of course, it is not supersymmetric. Moreover the infinite sequence of local and non-local conserved charges becomes certainly reduced, possibly only to a finite set. Consequently the associated symmetries have also been broken. In spite of the fact that the new integrable system has less symmetries, its solitonic solutions are stable in the Liapunov sense. The problem of stability of the solutions of the supersymmetric KdV system is also an open problem. Our result is a contribution in this sense.

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